

## ESTIMATES FOR OPERATORS IN MIXED WEIGHTED $L^p$ -SPACES

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**ABSTRACT.** A weighted Marcinkiewicz interpolation theorem is proved. If  $T$  is simultaneously of weak type  $(p_i, q_i)$ ,  $i = 0, 1$ ;  $1 \leq p_0 < p_1 \leq \infty$  and  $u, v$  certain weight functions, then  $T$  is bounded from  $L_v^p$  to  $L_u^q$  for  $0 < q < p$ ,  $p \geq 1$ . The result is applied to obtain weighted estimates for the Laplace and Fourier transform, as well as the Riesz potential.

1. In [8] we proved that if  $T$  is simultaneously of weak type  $(p_i, q_i)$ ,  $i = 0, 1$ ;  $1 \leq p_0 < p_1 < \infty$ ,  $1 \leq q_0, q_1 \leq \infty$ , then under certain conditions on weight functions  $u$  and  $v$ ,  $T$  is bounded from  $L_v^p$  to  $L_u^q$ , where  $1 \leq p \leq q \leq \infty$ . This is of course a weighted extension of the Marcinkiewicz interpolation theorem. The question remained open if this result also extends to the case  $q < p$ . It is the purpose of this paper to show that there are conditions on  $u$  and  $v$  for which this result extends to  $0 < q < p$ ,  $p \geq 1$ . As always, new interpolation results imply new results for a number of classical operators. Here we apply the results to the Laplace transform, complementing those obtained in [1, Theorem 2.4], the Fourier transform, extending results in [8, 10–12] and apply it to obtain a new inequality with applications in signal analysis [6]. Further we obtain a weighted extension of Sobolev's theorem.

In [9, Theorem 1.1] it was shown that the only bounded translation invariant operators from  $L^p$  to  $L^q$ ,  $q < p$ , are the zero operators. Our result (Theorem 3.3) shows that there are weights  $u$  and  $v$  for which there are such (nonzero) operators which map  $L_v^p$  to  $L_u^q$ ,  $0 < q < p$ ,  $p \geq 1$ .

In the next section we introduce notation, give a preliminary result and prove the main result (Theorem 2.2). §3 contains applications.

Throughout,  $p'$  denotes the conjugate index of  $p$  and is related to  $p$  by  $p + p' = pp'$  with  $p' = +\infty$  if  $p = 1$ , similarly for other letters. Further, constants are denoted by  $C$  and may be different at different appearances, but are always independent of the function in question.  $\mathbf{Z}$  denotes the set of integers.

2. Let  $u$  and  $v$  be nonnegative weight functions defined on  $\Omega \subseteq \mathbf{R}^n$ , and  $T$  a sublinear operator defined on Lebesgue measurable functions on  $\Omega$ . We denote by  $L_v^p \equiv L_v^p(\Omega)$ ,  $0 < p \leq \infty$ , the space of weighted measurable functions  $f$  for which

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$\|f\|_{v,p} \equiv \|vf\|_p$  is finite, where  $\|\cdot\|_p$  denotes the usual Lebesgue norm or metric depending on whether  $p \geq 1$  or  $0 < p < 1$ . We also write  $T \in [L_v^p, L_u^q]$ ,  $0 < p, q < \infty$ , if  $\|Tf\|_{u,q} \leq C\|f\|_{v,p}$ .

Let  $f$  be defined on  $(0, \infty)$ , then the Hardy operator  $P$  and its dual  $\bar{P}$  are defined by

$$(Pf)(x) = \int_0^x f(t) dt, \quad (\bar{P}f)(x) = \int_x^\infty f(t) dt, \quad x > 0.$$

It is well known [2, 4] that  $P \in [L_v^p, L_u^q]$ ,  $1 \leq p \leq q \leq \infty$ , if and only if

$$\sup_{s>0} \left( \int_0^\infty \chi_{(s,\infty)}(x) u(x)^q dx \right)^{1/q} \left( \int_0^\infty \chi_{(0,s)}(x) v(x)^{-p'} dx \right)^{1/p'} < \infty,$$

where here and in the sequel

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases}$$

is the characteristic function. The corresponding result for the dual operator also holds, only now the characteristic functions in the above integrals are interchanged.

In case  $0 < q < p$ ,  $1 \leq p \leq \infty$  the following result for the Hardy operator was proved by E. Sawyer [14, Theorem 3]:

**PROPOSITION 2.1.** (a) *Let  $0 < q < p$ ,  $1 \leq p \leq \infty$ , and  $1/r = 1/q - 1/p$ , then  $P \in [L_v^p, L_u^q]$ , if and only if*

$$(2.1) \quad \sup_{\dots < x_k < x_{k+1} < \dots} \left\{ \sum_k \left[ \left( \int_0^\infty \chi_k(x) u(x)^q dx \right)^{1/q} \times \left( \int_0^\infty \chi_{k-1}(x) v(x)^{-p'} dx \right)^{1/p'} \right]^r \right\}^{1/r} < \infty,$$

where  $\chi_k(x) = \chi_{(x_k, x_{k+1})}(x)$  and the supremum is taken over all positive increasing sequences and  $k \in \mathbf{Z}$ .

(b) *Under the same conditions on  $p$  and  $q$ ,  $\bar{P} \in [L_v^p, L_u^q]$  if and only if*

$$(2.2) \quad \sup_{\dots < x_k < x_{k+1} < \dots} \left\{ \sum_k \left[ \left( \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right)^{1/q} \times \left( \int_0^\infty \chi_k(x) v(x)^{-p'} dx \right)^{1/p'} \right]^r \right\}^{1/r} < \infty.$$

If  $p = 1$  the second integral is interpreted as usual to be the supremum of  $1/v$  over the appropriate range.

**PROOF.** The proof of (a) is given in [14]. The proof for the dual operator is quite similar and is given here only for completeness.

It clearly suffices to prove the result for  $f \geq 0$ . If  $\bar{P} \in [L_v^p, L_u^q]$ , fix a positive increasing sequence  $\{x_k\}_{k \in \mathbf{Z}}$  and a given sequence of nonnegative numbers  $\{a_k\}$

and define

$$f(x) = \sum_k a_k \chi_k(x) v(x)^{-p'};$$

then

$$\begin{aligned} \left( \sum_k a_k^p \int_0^\infty \chi_k(x) v(x)^{-p'} dx \right)^{q/p} &= \left( \int_0^\infty [v(x)f(x)]^p dx \right)^{q/p} \\ &\geq C \int_0^\infty [u(x)(\bar{P}f)(x)]^q dx \\ &= C \sum_k \int_{x_{k-1}}^{x_k} u(x)^q \left( \int_x^\infty f(t) dt \right)^q dx \\ &\geq C \sum_k \left( \int_{x_{k-1}}^{x_k} u(x)^q dx \right) \left( \int_{x_k}^\infty f(t) dt \right)^q \\ &= C \sum_k \left[ a_k \int_0^\infty \chi_k(t) v(t)^{-p'} dt \right]^q \left( \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right) \\ &= C \sum_k \left( a_k^p \int_0^\infty \chi_k(t) v(t)^{-p'} dt \right)^{q/p} \left( \int_0^\infty \chi_k(t) v(t)^{-p'} dt \right)^{1/p'} \\ &\quad \times \left( \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right) \end{aligned}$$

all sequences  $\{a_k\}$  of nonnegative numbers. But the dual of  $l^{p/q}$ ,  $p \geq 1$  is  $l^{(p/q)'};$  hence,

$$b_k \equiv \left[ \left( \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right)^{1/q} \left( \int_0^\infty \chi_k(t) v(t)^{-p'} dt \right)^{1/p'} \right]^q$$

is in  $l^{(p/q)'}$ . But  $(p/q)' = p/(p-q)$ , so that

$$\sum_k \left[ \left( \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right)^{1/q} \left( \int_0^\infty \chi_k(x) v(x)^{-p'} dx \right)^{1/p'} \right]^r < \infty,$$

which proves that (2.2) is necessary.

Conversely, fix  $f \geq 0$  in  $L_v^p$  and choose  $\{x_k\}$  such that

$$\int_{x_k}^\infty f(t) dt = 2^{-k}$$

for all  $k \in \mathbb{Z}$  for which

$$2^{-k} < \int_0^\infty f(t) dt.$$

Then Hölder's inequality applied to the integral and the sum shows that

$$\begin{aligned}
 & \left[ \int_0^\infty [u(x)(\bar{P}f)(x)]^q dx \right]^{p/q} \leq \left[ \sum_k \int_{x_{k-1}}^{x_k} u(x)^q dx \left( \int_{x_{k-1}}^\infty f(t) dt \right)^q \right]^{p/q} \\
 & = \left[ \sum_k 2^{(-k+1)q} \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right]^{p/q} \\
 & = 2^{2p} \left[ \sum_k \left( \int_{x_k}^{x_{k+1}} f(t) dt \right)^q \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right]^{p/q} \\
 & \leq 2^{2p} \left[ \sum_k \left( \int_{x_k}^{x_{k+1}} [v(t)f(t)]^p dt \right)^{q/p} \left( \int_{x_k}^{x_{k+1}} v(t)^{-p'} dt \right)^{q/p'} \right. \\
 & \quad \left. \times \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right]^{p/q} \\
 & \leq 2^{2p} \sum_k \int_{x_k}^{x_{k+1}} [v(t)f(t)]^p dt \left[ \sum_k \left[ \left( \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right)^{1/q} \right. \right. \\
 & \quad \left. \left. \times \left( \int_0^\infty \chi_k(t) v(t)^{-p'} dt \right)^{1/p'} \right]^r \right]^{p/r} \\
 & \leq 2^{2p} \|f\|_{v,p}^p \left[ \sum_k \left[ \left( \int_0^\infty \chi_{k-1}(x) u(x)^q dx \right)^{1/q} \left( \int_0^\infty \chi_k(t) v(t)^{-p'} dt \right)^{1/p'} \right]^r \right]^{p/r},
 \end{aligned}$$

which proves the result for  $p < \infty$ . If  $p = \infty$  the argument is similar and hence omitted, while in the case  $p = 1$  one modifies the argument as in [14].

Note that if  $C$  denotes the supremum of (2.2), then this shows that  $\|\bar{P}f\|_{u,q} \leq 4C\|f\|_{v,p}$  and similarly for  $P$ . We shall make use of this observation in §3.

Before proving the main results, we recall the concept of rearrangement and radial rearrangement of a function.

Let  $f$  be Lebesgue measurable on  $\Omega \subseteq \mathbf{R}^n$  and let  $m$  denote Lebesgue measure. The equimeasurable decreasing rearrangement of  $f$  is defined by

$$f^*(t) = \inf \{ y > 0 : m(\{x \in \Omega : |f(x)| > y\}) \leq t \}.$$

If  $x \in \mathbf{R}^n$ , let  $t = |x|$  and  $\theta_n$  the volume of the unit sphere in  $\mathbf{R}^n$ , then the radial rearrangement of  $f$  is defined by  $f^\circ(t) = f^*(\theta_n t^n)$ . Since  $n\theta_n$  is the surface area of the unit  $n$ -sphere, it follows that

$$\int_{\mathbf{R}^n} |f(x)| dx = n\theta_n \int_0^\infty f^\circ(t) t^{n-1} dt = \int_{\mathbf{R}^n} f^\circ(|x|) dx.$$

We recall further that a sublinear operator  $T$  defined on simple functions is of weak type  $(p, q)$ ,  $1 \leq p, q \leq \infty$ , if and only if  $\sup_{t>0} t^{1/q} (Tf)^*(t) \leq C\|f\|_p$ . Strong type  $(p, q)$  means of course that  $T \in [L^p, L^q]$ . We now prove a result corresponding to [8, Theorem 2.4] but with  $0 < q < p, p \geq 1$ .

**THEOREM 2.2.** *Suppose  $T$  is sublinear of weak type  $(p_i, q_i)$ ,  $i = 0, 1$ ;  $1 \leq p_0 < p_1 < \infty$ ,  $1 \leq q_0, q_1 \leq \infty$ . Let  $\lambda = 1/q_0 - 1/q_1 \neq 0$ ,  $\bar{\lambda} = 1/p_0 - 1/p_1$  and for given real*

increasing sequence  $\{x_k\}_{k \in \mathbb{Z}}$  define

$$\Lambda_k(s) = \begin{cases} 1 & \text{if } (\theta_n^{1/n} x_k)^\lambda \leq (s \theta_n^{1/n})^\lambda \leq (\theta_n^{1/n} x_{k+1})^\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

If  $0 < q < p$ ,  $1 \leq p \leq \infty$ ,  $1/r = 1/q - 1/p$  and  $U, V$  radial on  $\mathbb{R}^n$  satisfying

$$(2.3) \quad \sup \left\{ \sum_k \left[ \left( \int_0^\infty \Lambda_k(s) [U(s) s^{-n/q_0}]^q s^{n-1} ds \right)^{1/q} \times \left( \int_0^\infty \chi_{k-1}(s) [V(s) s^{n(1/p-1/p_0)}]^{-p'} s^{-1} ds \right)^{1/p'} \right]^r \right\}^{1/r} < \infty,$$

(2.4)

$$\sup \left\{ \sum_k \left[ \left( \int_0^\infty \Lambda_{k-1}(s) [U(s) s^{-n/q_1}]^q s^{n-1} ds \right)^{1/q} \times \left( \int_0^\infty \chi_k(s) [V(s) s^{n(1/p-1/p_1)}]^{-p'} s^{-1} ds \right)^{1/p'} \right]^r \right\}^{1/r} < \infty,$$

where the supremum is taken over all positive increasing sequences  $\{x_k\}$ , then

$$\left\{ \int_{\mathbb{R}^n} |U(|x|)(Tf)^\circ(|x|)|^q dx \right\}^{1/q} \leq C \left\{ \int_{\mathbb{R}^n} |V(|x|)f^\circ(|x|)|^p dx \right\}^{1/p}.$$

PROOF. The hypotheses on the operator are satisfied [5, Theorems 8, 9] if and only if

$$(Tf)^*(t) \leq C \left[ t^{-1/q_0} \int_0^{t^{\lambda/\bar{\lambda}}} s^{1/p_0-1} f^*(s) ds + t^{-1/q_1} \int_{t^{\lambda/\bar{\lambda}}}^\infty s^{1/p_1-1} f^*(s) ds \right].$$

On replacing  $t$  by  $\theta_n t^n$  and  $s$  by  $\theta_n s^n$  in this estimate one obtains

$$(Tf)^\circ(t) \leq C \left[ t^{-n/q_0} \int_0^{\kappa t^{\lambda/\bar{\lambda}}} s^{n/p_0-1} f^\circ(s) ds + t^{-n/q_1} \int_{\kappa t^{\lambda/\bar{\lambda}}}^\infty s^{n/p_1-1} f^\circ(s) ds \right],$$

where  $\kappa = \theta_n^{(\lambda-\bar{\lambda})/(n\bar{\lambda})}$ . Now multiply the inequality by  $U(t)t^{(n-1)/q}$  and integrate, then by Minkowski's inequality if  $q \geq 1$  and directly if  $0 < q < 1$

$$\begin{aligned} & \left\{ \int_0^\infty [U(t)(Tf)^\circ(t)]^q t^{n-1} dt \right\}^{1/q} \\ & \leq C \left\{ \int_0^\infty U(t)^q t^{n-1-nq/q_0} \left( \int_0^{\kappa t^{\lambda/\bar{\lambda}}} s^{n/p_0-1} f^\circ(s) ds \right)^q dt \right\}^{1/q} \\ & \quad + C \left\{ \int_0^\infty U(t)^q t^{n-1-nq/q_0} \left( \int_{\kappa t^{\lambda/\bar{\lambda}}}^\infty s^{n/p_1-1} f^\circ(s) ds \right)^q dt \right\}^{1/q} \\ & \equiv C[J_0 + J_1], \end{aligned}$$

respectively. To estimate  $J_0$  let  $x = \kappa t^{\lambda/\bar{\lambda}}$ , then  $t = (x/\kappa)^{\bar{\lambda}/\lambda}$  and hence by Proposition 2.1(a)

$$\begin{aligned} J_0 &= C \left\{ \int_0^\infty U((x/\kappa)^{\bar{\lambda}/\lambda})^q x^{\bar{\lambda}n(1-q/q_0)/\lambda-1} \left( \int_0^x s^{n/p_0-1} f^\circ(s) ds \right)^q dx \right\}^{1/q} \\ &\leq C \left\{ \int_0^\infty [V(t)f^\circ(t)]^p t^{n-1} dt \right\}^{1/p} \end{aligned}$$

if and only if

$$(2.5) \quad \sup \left\{ \sum_k \left[ \left( \int_{x_k}^{x_{k+1}} \left[ U((x/\kappa)^{\bar{\lambda}/\lambda}) \right]^q x^{\bar{\lambda}n(1-q/q_0)/\lambda-1} dx \right)^{1/q} \right. \right. \\ \left. \left. \times \left( \int_{x_{k-1}}^{x_k} \left[ x^{-n/p_0+1} V(x) x^{(n-1)/p} \right]^{-p'} dx \right)^{1/p'} \right]^r \right\}^{1/r} < \infty.$$

Let  $s = (x/\kappa)^{\bar{\lambda}/\lambda}$ ; then apart from a constant, the first integral in (2.5) becomes

$$\int U(s)^q s^{-nq/q_0+n-1} ds$$

with upper limit of integration  $(x_{k+1}/\kappa)^{\bar{\lambda}/\lambda}$  and lower limit  $(x_k/\kappa)^{\bar{\lambda}/\lambda}$  provided  $\lambda > 0$ . The limits are reversed if  $\lambda < 0$ . Since the second integral of (2.5) is equal to

$$\int_{x_{k-1}}^{x_k} \left[ V(x) x^{n(1/p-1/p_0)} \right]^{-p'} x^{-1} dx,$$

it follows that (2.5) is equivalent to (2.3).

The estimate for  $J_1$  is similar. Again let  $x = \kappa t^{\lambda/\bar{\lambda}}$ ; then by Proposition 2.1(b)

$$J_1 = C \left\{ \int_0^\infty \left[ U((x/\kappa)^{\bar{\lambda}/\lambda}) \right]^q x^{\bar{\lambda}n(1-q/q_1)/\lambda-1} \left( \int_x^\infty s^{n/p_1-1} f^\circ(s) ds \right)^q dx \right\}^{1/q} \\ \leq C \left\{ \int_0^\infty [V(t) f^\circ(t)]^p t^{n-1} dt \right\}^{1/p}$$

if and only if

$$(2.6) \quad \sup \left\{ \sum_k \left[ \left( \int_{x_{k-1}}^{x_k} \left[ U((x/\kappa)^{\bar{\lambda}/\lambda}) \right]^q x^{\bar{\lambda}n(1-q/q_1)/\lambda-1} dx \right)^{1/q} \right. \right. \\ \left. \left. \times \left( \int_{x_k}^{x_{k+1}} \left[ x^{-n/p_1+1} V(x) x^{(n-1)/p} \right]^{-p'} dx \right)^{1/p'} \right]^r \right\}^{1/r} < \infty.$$

Again if  $s = (x/\kappa)^{\bar{\lambda}/\lambda}$ , then a routine calculation shows that (2.6) is equivalent to (2.4).

From these two estimates we obtain

$$\left( \int_{\mathbf{R}^n} [U(|x|)(Tf)^\circ(|x|)]^q dx \right)^{1/q} = \left( n\theta_n \int_0^\infty [U(t)(Tf)^\circ(t)]^q t^{n-1} dt \right)^{1/q} \\ \leq C \left( \int_0^\infty [V(t) f^\circ(t)]^p t^{n-1} dt \right)^{1/p} = C \left( \int_{\mathbf{R}^n} [V(|x|) f^\circ(|x|)]^p dx \right)^{1/p}$$

and hence the result.

Note that the proof shows that the constant  $C$  is a multiple of the sum of the suprema (2.3) and (2.4).

**COROLLARY 2.3.** *Let  $T, U$  and  $V$  satisfy the conditions of Theorem 2.2, then for  $0 < q < p, 1 \leq p \leq \infty, \|(Tf)^\circ\|_{U,q} \leq C\|f^\circ\|_{V,p}$ .*

*Furthermore, if  $u$  and  $v$  are such that  $u^\circ = U$  and  $(1/v)^\circ = 1/V$ , then  $T \in [L_v^p, L_u^q]$ ,  $0 < q < p, 1 \leq p \leq \infty$ . If  $p < \infty$  the operator  $T$  extends uniquely to all of  $L_v^p$  with the same bound.*

**PROOF.** Clearly we only need to prove the second statement of the corollary. By Theorem 2.2

$$\begin{aligned} \|Tf\|_{u,q} &\leq C \left( \int_0^\infty [u^\circ(t)(Tf)^\circ(t)]^q t^{n-1} dt \right)^{1/q} \leq C \left( \int_{\mathbf{R}^n} [U(|x|)(Tf)^\circ(|x|)]^q dx \right)^{1/q} \\ &\leq C \left( \int_{\mathbf{R}^n} [V(|x|)f^\circ(|x|)]^p dx \right)^{1/p} = C \left( \int_{\mathbf{R}^n} [1/(1/v)^\circ(|x|)f^\circ(|x|)]^p dx \right)^{1/p} \\ &\leq C \left( \int_{\mathbf{R}^n} |v(x)f(x)|^p dx \right)^{1/p}, \end{aligned}$$

where the first inequality follows from the properties of rearrangements. The last inequality follows from the integral analogue of [7, Theorem 368] by approximating  $f$  and  $v$  by appropriate simple functions and then use Lebesgue's theorem of monotone convergence.

If  $p < \infty$ , simple functions are dense in the Banach space  $L_v^p$ . Since  $f \rightarrow Tf$  is continuous from a dense subspace of  $L_v^p$  to  $L_u^q$ ,  $T$  has a unique extension to all of  $L_v^p$  with the same operator bound.

**3.** In our first application we consider a special class of operators of which the Fourier transform and the Laplace transform are special cases. For simplicity we consider the one-dimensional case.

**THEOREM 3.1.** *Let  $T$  be an operator of weak type  $(1, \infty)$  and (2.2) defined on  $L_v^p(\mathbf{R})$ ,  $1 \leq p < \infty$ . If  $0 < q < p, 1/r = 1/q - 1/p$  and*

$$(3.1) \quad \sup \left\{ \sum_k \left[ \left( \int_{E_k} u^\circ(s)^q ds \right)^{1/q} \left( \int_0^\infty \chi_{k-1}(s) (1/v)^\circ(s)^{p'} ds \right)^{1/p'} \right]^r \right\}^{1/r},$$

$$(3.2) \quad \sup \left\{ \sum_k \left[ \left( \int_{E_{k-1}} [u^\circ(s)s^{-1/2}]^q ds \right)^{1/q} \left( \int_0^\infty \chi_k(s) [(1/v)^\circ(s)s^{-1/2}]^{p'} ds \right)^{1/p'} \right]^r \right\}^{1/r}$$

*are finite, where  $E_k = (1/(4x_{k+1}), 1/(4x_k))$  and the supremum are taken over all increasing sequences  $\{x_k\}_{k \in \mathbf{Z}}$ , then  $T \in [L_v^p, L_u^q]$ .*

If  $p = \infty$  the result still holds; however, now  $T$  is assumed to be defined on simple functions only.

**PROOF.** If  $n = 1, \theta_n = 2$ , also,  $\lambda = -1/2, \lambda = 1/2$  and  $\Lambda_k(s) = 1$  if  $s \in E_k$  and zero otherwise, in Theorem 2.2, then the result now follows from Theorem 2.2 and Corollary 2.3.

As implied above, Theorem 3.1 applies specifically to the Fourier transform and Laplace transform

$$(\mathcal{F}f)(x) = \int_{-\infty}^{\infty} e^{ixy} f(y) dy, \quad x \in \mathbf{R}; \quad (\mathcal{L}f)(x) = \int_0^{\infty} e^{-xy} f(y) dy, \quad x > 0,$$

respectively. For the Laplace transform the radial rearrangements in (3.2) are to be interpreted as the increasing rearrangements.

Note that (3.2) is dominated by

$$(3.3) \quad \sup \left\{ \sum_k \left[ \left( \int_{E_{k-1}} [u^\circ(s)]^q ds \right)^{1/q} \left( \int_0^\infty \chi_k(s) [(1/v)^\circ(s)]^{p'} ds \right)^{1/p'} \right]^r \right\}^{1/r};$$

however, (3.1) does in general not imply (3.3).

Now we show that with  $T = \mathcal{L}$  a partial converse of Theorem 3.1 can be obtained.

**PROPOSITION 3.2.** *Let  $0 < q < p$ ,  $1 < p < \infty$ ,  $u$  decreasing and  $v$  increasing on  $(0, \infty)$ , then  $\mathcal{L} \in [L_v^p, L_u^q]$  implies (3.1).*

**PROOF.** If  $u$  is decreasing and  $v$  increasing, then  $u = u^\circ$  and  $1/v = (1/v)^\circ$ .

Fix a positive increasing sequence  $\cdots < x_k < x_{k+1} < \cdots$  and let  $\{a_k\}$  be a positive sequence. Define  $f$  by

$$f(x) = \sum_k a_k \chi_{k-1}(x) v(x)^{-p'}, \quad \chi_{k-1}(x) = \chi_{(x_{k-1}, x_k)}(x),$$

then

$$\begin{aligned} \left( \sum_k a_k^p \int_{x_{k-1}}^{x_k} v(x)^{-p'} dx \right)^{1/p} &= \left( \sum_k \int_{x_{k-1}}^{x_k} v(x)^p \left| \sum_k a_k \chi_{k-1}(x) v(x)^{-p'} \right|^p dx \right)^{1/p} \\ &= \left( \int_0^\infty |v(x) f(x)|^p dx \right)^{1/p} \geq C \left( \int_0^\infty |u(x) (\mathcal{L}f)(x)|^q dx \right)^{1/q} \\ &= C \left( \int_0^\infty u(x)^q \left[ \sum_j a_j \int_{x_{j-1}}^{x_j} e^{-xy} v(y)^{-p'} dy \right]^q dx \right)^{1/q} \\ &= C \left( \sum_k \int_{E_k} u(x)^q \left[ \sum_j a_j \int_{x_{j-1}}^{x_j} e^{-xy} v(y)^{-p'} dy \right]^q dx \right)^{1/q} \\ &\geq C \left( \sum_k \int_{E_k} u(x)^q a_k^q \left[ \int_{x_{k-1}}^{x_k} e^{-xy} v(y)^{-p'} dy \right]^q dx \right)^{1/q} \\ &\geq C \cdot e^{-1/4} \left( \sum_k a_k^q \left[ \left( \int_{E_k} u(x)^q dx \right)^{1/q} \left( \int_0^\infty \chi_{k-1}(y) v(y)^{-p'} dy \right) \right]^q \right)^{1/q}. \end{aligned}$$

Let  $A_k = a_k^q \left( \int_0^\infty \chi_{k-1}(y) v(y)^{-p'} dy \right)^{q/p}$ ; then this shows that

$$\left( \sum_k A_k^{p/q} \right)^{q/p} \geq C \sum_k A_k B_k,$$



where

$$B_k = \left[ \left( \int_{E_k} u(x)^q dx \right)^{1/q} \left( \int_0^\infty \chi_{k-1}(y) v(y)^{-p'} dy \right)^{1/p'} \right]^q.$$

Since  $\{A_k\} \in l^{(p/q)}$  it follows that  $\{B_k\} \in l^{(p/q)'}$ . But  $(p/q)' = p/(p-q) = r/q$  so that

$$\sum_k \left[ \left( \int_{E_k} u(x)^q dx \right)^{1/q} \left( \int_0^\infty \chi_{k-1}(y) v(y)^{-p'} dy \right)^{1/p'} \right]^r < \infty$$

and this implies the result.

The next result has applications in the theory of signal analysis and may be compared to [6, Theorem 3]. (I am grateful to J. J. Benedetto for providing that preprint.)

**THEOREM 3.3.** *Let  $1 \leq p, q \leq \infty$  and  $\alpha, \beta$  real numbers. If*

(i)  $1 \leq p \leq q \leq \infty$  and  $0 \leq 1/p' - 1/q < \theta < 1/p'$ , or

(ii)  $1 \leq q < p < \infty$  and  $1/p' < \theta < \infty$ ,

then

$$(3.4) \quad \left( \int_{-\alpha}^\beta |(\mathcal{F}f)(y)|^q dy \right)^{1/q} \leq C(\alpha + \beta)^{1/q + \theta - 1/p'} \| |x|^\theta f \|_p.$$

**PROOF.** (i) If  $1 \leq p \leq q \leq \infty$ , then by [8, Theorem 3.1] (see also [3, 10–12]),  $\mathcal{F} \in [L_v^p, L_u^q]$  provided

$$(3.5) \quad \sup_{s>0} \left( \int_0^{1/(2s)} u^\circ(x)^q dx \right)^{1/q} \left( \int_0^{s/2} (1/v)^\circ(x)^{p'} dx \right)^{1/p'} < \infty$$

with the usual modification if  $p = 1$  and/or  $q = \infty$ . Specifically, if  $u(x) = \chi_{(-\alpha, \beta)}(x)$  and  $v(x) = |x|^\theta$ ,  $\theta \geq 0$ ,  $u^\circ(x) = \chi_{(-(\alpha+\beta)/2, (\alpha+\beta)/2)}(x)$ ,  $(1/v)^\circ(x) = |x|^{-\theta}$ . Therefore (3.4) holds if (3.5) is dominated by  $(\alpha + \beta)^{1/q + \theta - 1/p'}$ . Let  $1/p' - 1/q < \theta < 1/p'$ , if  $1/(2s) \leq (\alpha + \beta)/2$ , then (3.5) is dominated by

$$1/(2s)^{1/q} (s/2)^{-\theta + 1/p'} \leq C(\alpha + \beta)^{1/q + \theta - 1/p'}.$$

If  $\alpha + \beta < 1/s$ , then (3.5) is dominated by

$$(\alpha + \beta)^{1/q} s^{-\theta + 1/p'} \leq (\alpha + \beta)^{1/q + \theta - 1/p'}$$

which proves this case.

(ii) If  $1 \leq q < p < \infty$  and  $1/p' < \theta$  we apply Theorem 3.1. For fixed positive increasing sequence  $\{x_k\}$ , it suffices to show that

$$(3.6) \quad \left\{ \sum_k \left[ \left( \int_{E_k} \chi_{(0, (\alpha+\beta)/2)}(x) dx \right)^{1/q} \left( \int_0^\infty \chi_{k-1}(x) x^{-\theta p'} dx \right)^{1/p'} \right]^r \right\}^{1/r},$$

and by (3.3) the same sum with  $\chi_{k-1}$  replaced by  $\chi_{k+1}$  are both bounded by  $C \cdot (\alpha + \beta)^{1/q + \theta - 1/p'}$  (cf. the remark following the proof of Theorem 2.2). Here again  $1/r = 1/q - 1/p$ .

If  $\alpha + \beta \leq 1/(2x_{k+1})$  the first integral in (3.6) is zero and hence so are all terms of the sums for which  $x_{k+1} \leq 1/(2(\alpha + \beta))$ . We consider therefore only those  $k$  for which  $x_{k+1} > 1/(2(\alpha + \beta))$ . The first integral is clearly dominated by  $(\alpha + \beta)/2$  and since  $r/p' > 1$ , (3.6) is dominated by

$$\begin{aligned} C(\alpha + \beta)^{1/q} & \left\{ \sum_{\substack{k \\ x_{k+1} > [2(\alpha + \beta)]^{-1}}} \int_{x_{k-1}}^{x_k} x^{-\theta p'} dx \right\}^{1/p'} \\ & \leq C(\alpha + \beta)^{1/q} \left( \int_{[2(\alpha + \beta)]^{-1}}^{\infty} x^{-\theta p'} dx \right)^{1/p'} \\ & = C(\alpha + \beta)^{1/q + \theta - 1/p'}. \end{aligned}$$

If  $\chi_{k-1}$  in (3.6) is replaced by  $\chi_{k+1}$  then the same argument shows that this sum is dominated by

$$C(\alpha + \beta)^{1/q} \left[ \sum_{\substack{k \\ x_{k+1} > [2(\alpha + \beta)]^{-1}}} \int_{x_{k+1}}^{x_{k+2}} x^{-\theta p'} dx \right]^{1/p'} = C(\alpha + \beta)^{1/q + \theta - 1/p'}.$$

This proves the result.

In this last example, we consider the Riesz potential  $I_\alpha$ , defined by

$$(I_\alpha f)(x) = C_n \int_{\mathbf{R}^n} |x - y|^{\alpha - n} f(y) dt, \quad 0 < \alpha < n.$$

If  $1 \leq p \leq q \leq \infty$ ,  $p < \infty$ , then [4, Theorem 3.4] shows that  $I_\alpha \in [L_v^p(\mathbf{R}^n), L_u^q(\mathbf{R}^n)]$ . On the other hand if  $q < p$  [9, Theorem 1.1] shows that there is no nonzero translation invariant operator in  $[L^p, L^q]$ . In the result below we prove that for  $0 < q < p$ ,  $p \geq 1$ , there are weights  $u, v$  for which such operators are in  $[L_v^p, L_u^q]$ .

**THEOREM 3.4.** *Suppose  $u$  and  $v$  are nonnegative functions on  $\mathbf{R}^n$ ,  $U = u^\circ$ ,  $1/V = (1/v)^\circ$ ,  $0 < q < p$ ,  $1 \leq p < \infty$  and  $1/r = 1/q - 1/p$ . If for some  $s$ ,  $1 < s < n/\alpha$ ,*

$$\begin{aligned} \sup \left\{ \sum_k \left[ \left( \int_0^\infty \chi_k(x) [U(x)x^{\alpha-n}]^q x^{n-1} dx \right)^{1/q} \right. \right. \\ \left. \left. \times \left( \int_0^\infty \chi_{k-1}(x) V(x)^{-p'} x^{n-1} dx \right)^{1/p'} \right]^r \right\}^{1/r} \end{aligned}$$

and

$$\begin{aligned} \sup \left\{ \sum_k \left[ \left( \int_0^\infty \chi_{k-1}(x) [U(x)x^{\alpha-n/s}]^q x^{n-1} dx \right)^{1/q} \right. \right. \\ \left. \left. \times \left( \int_0^\infty \chi_k(x) [V(x)x^{n/s}]^{-p'} x^{n-1} dx \right)^{1/p'} \right]^r \right\}^{1/r} \end{aligned}$$

are finite, where the supremum is taken over all positive increasing sequences  $\{x_k\}_{k \in \mathbb{Z}}$ , then  $I_\alpha \in [L_v^p(\mathbb{R}^n), L_u^q(\mathbb{R}^n)]$ .

PROOF. It is well known [15, V, §1.2] that  $I_\alpha$  is of weak type  $(1, 1/(1 - \alpha/n))$ , and  $(s, 1/(1/s - \alpha/n))$ ,  $1 < s < n/\alpha$ , so that, in the notation of Theorem 2.2,  $p_0 = 1$ ,  $p_1 = s$ ,  $q_0 = 1/(1 - \alpha/n)$ ,  $\lambda = \bar{\lambda} = 1/s'$  and  $\Lambda_k(x) = X_k(x)$ . Therefore the conditions (2.3) and (2.4) are satisfied by the assumption of the theorem and hence the result follows from Theorem 2.2 and Corollary 2.3.

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